under λ is equivalent to π. If π occurs at least once we say π is a discrete component of λ. Moreover we say that π occurs only a finite number of times if there exists an integer $m \geq 1$ such that it is impossible to choose $\mathfrak{H}_i$ $(1 \leq i \leq m)$ with the above properties.

The following result is an immediate consequence of Lemma 1 and Theorem 1.

THEOREM 2. *Let* Γ *be a discrete subgroup of* G *of type* III *and* Γ' *a subgroup of finite index in* Γ. *Put* $\Gamma_0 = \bigcap_{\gamma \in \Gamma} \gamma\Gamma'\gamma^{-1}$ *and suppose that* $N \cap \Gamma/N \cap \Gamma_0$ *is finite. Then every discrete irreducible component of the representation* λ *of* G *on* $L_2(G/\Gamma')$ *occurs only a finite number of times in* λ.

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[1] See Séminaire H. Cartan, 1957/58, *Exposé 8*, pp. 8–10.

[2] *Ann. of Math.*, **50**, 525 (1949).

[3] This condition was suggested by Godement.

TIME OPTIMAL CONTROL SYSTEMS*

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1. *Introduction.*—It has been an intuitive assumption for some time that if a control system is being operated from a limited source of power then the system can be moved from one state to another in the shortest time by at all times utilizing properly all available power. This hypothesis is called the "bang-bang principle." Bushaw accepted this hypothesis and in 1952 showed for some simple systems with one degree of freedom that of all bang-bang systems (that is, systems which at all times utilize maximum power) there is one that is optimal.[1] In 1953 I made the observation that the best of all bang-bang systems, if it exists, is then the best of all systems operating from the same power source.[2] More recently fairly general results have been obtained by Bellman, Glicksberg, and Gross[3] and later (but seemingly independently) by Krasovskii[4] and Gamkrelidze.[5] At the 1958 International Congress of Mathematicians in Edinburgh, L. S. Pontryagin announced a "maximum principle" which is the beginning of an even more general theory.

We confine ourselves here to the time optimal problem for control systems which are linear in the sense that the elements being controlled are linear and as a function of time the control enters linearly. The differential equation for such systems is

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t), \tag{1}$$

where x and f are n-dimensional vector functions (x(t) is the state of the system at time t), A is an $(n \times n)$ matrix function, and B is an $(n \times r)$ matrix function. Thus (1) represents the system of differential equations

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{k=1}^{r} b_{ik}(t)u_k(t) + f_i(t),\ i = 1, \ldots, n.$$

Our ability to control the system lies in the freedom we have to choose the "steering" function u. We assume that the admissible steering functions are piecewise continuous (or measurable) and have components less than 1 in absolute value ($|u_i(t)| \leq 1$). Given an initial state x_0 and a moving particle z(t), the problem of time optimal control is to hit the particle in minimum time. Let x(t, u) be the solution of (1) satisfying $x(0) = x_0$. An admissible steering function u^* is *optimal* if for some $t^* > 0$, $x(t^*, u^*) = z(t^*)$ and if $x(t, u) \neq z(t)$ for $0 < t < t^*$ and all admissible u.

Bellman, Glicksberg, and Gross[3] considered the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2}$$

and restricted themselves to the problem of starting at x_0 and reaching the origin in minimum time. The $(n \times n)$ matrix A is constant and its characteristic roots were assumed to have negative real parts. B was assumed to be a constant nonsingular $(n \times n)$ matrix. For some of the simplest examples of control systems the matrix B is singular, and this restriction on B is much too severe. They prove the existence of an optimal steering function, and the form for an optimal steering function is given in the proof. However, the form given for an optimal steering function does not, in general, imply that there is a bang-bang optimal steering function. Gamkrelidze[5] considered the same problem, removed the restriction that B be nonsingular, and showed for systems which are later in this paper called "normal" the existence and uniqueness of an optimal steering function. The form of the optimal steering is the same as that given by Bellman, Glicksberg, and Gross, and in this case one can conclude that the optimal steering is bang-bang. Krasovskii[4] studied the more general control system (1) and the more general control problem of hitting a moving particle. Using results of Krein on the L-problem in abstract spaces, he proved the existence of an optimal steering function for systems which we call "proper" control systems. If Krein's results are to be used without modification, the restriction to proper control systems seems to be necessary. Krasovskii states also that the optimal steering function is unique and simple examples show this to be false. Thus to date the most general bang-bang principle has been proved by Gamkrelidze.

For the more general control system (1) we show essentially that anything that can be done by an admissible steering function can also be accomplished by using bang-bang steering. This extends our result[2] and at the same time establishes the bang-bang principle for all control systems where the controlled elements are linear. This does not mean that all optimal steering functions are bang-bang. For some systems the objective can be reached in minimal time using a steering function which, during part of the time, has some zero components. We state a number of results for proper and for normal control systems which show the significance of these classifications. As in the special problem considered by Gamkrelidze, the more general normal systems have unique optimal steering functions, and in this case we have a true bang-bang principle: the only way to reach the objective in minimum time is to use the maximum available power all of the time. In Theorem 5 we give a result which should be of importance in the synthesis problem, which is the problem of determining the optimal steering u^* as a function of the state of the system. This result shows that for some systems optimal steering can be determined by what amounts to running the system backwards. This idea gives, for instance, a much simplified solution of the example solved in the paper of reference 3.

2. The General Problem.—The problem described in the introduction for the system (1) of hitting a moving particle in minimum time will be called the *general problem*. For the control system (1) the state $x(t, u)$ of the system at time t is given by

$$x(t, u) = X(t)x_0 + X(t)\int_0^t Y(\tau)u(\tau)d\tau + X(t)\int_0^t X^{-1}(\tau)f(\tau)d\tau. \tag{3}$$

$X(t)$ is the principal matrix solution of $\dot{X}(t) = A(t)X(t)$, and $Y(\tau) = X^{-1}(\tau)B(\tau)$. We want at some time t to have $x(t) = z(t)$; i.e., to have

$$w(t) = \int_0^t Y(\tau)u(\tau)d\tau, \tag{4}$$

where $w(t) = X^{-1}(t)z(t) - x_0 - \int_0^t X^{-1}(\tau)f(\tau)d\tau$. We assume throughout that $A(t)$, $B(t)$, and $f(t)$ are continuous for $0 \leq t < \infty$. The following Lemma states that anything that can be done by an admissible steering function can also be done by a bang-bang function. The set of admissible steering functions is the set Ω and the set of bang-bang steering functions is the set Ω^0. The set $K(t)$ is related to the set of all states that can be reached in time t by an admissible steering function. $K^0(t)$ is similarly related to the set of states that can be reached in time t by bang-bang steering functions.[6]

LEMMA 1. *Let Ω be the set of all r-dimensional vector functions measurable on $[0, t]$ with $|u_i(\tau)| \leq 1$. Let Ω^0 be the subset of functions in Ω with $|u_i(\tau)| = 1$. Let $Y(\tau)$ be any $(n \times r)$ matrix function in $L^2([0, t])$. Define*

$$K(t) = \left\{\int_0^t Y(\tau)u(\tau)d\tau; \quad u \in \Omega\right\}$$

and

$$K^0(t) = \left\{\int_0^t Y(\tau)u^0(\tau)d\tau; \quad u^0 \in \Omega^0\right\}.$$

Then $K^0(t)$ is closed, and $K^0(t) = K(t)$.

As a direct consequence of the Lemma we obtain an extension of the result in the papeı of reference and a general bang-bang principle.

THEOREM 1. *If of all bang-bang steering functions there is an optimal one (relative to Ω^0), then it is optimal (relative to Ω).*

THEOREM 2. *If there is an optimal steering function (in Ω) then there is always a bang-bang steering function (in Ω^0) that is optimal.*

From Lemma 1 it is also not difficult to show that

THEOREM 3. *If for the general problem there is a steering function u in Ω such that $x(t, u) = z(t)$ for some $t > 0$, then there is an optimal steering function in Ω. Moreover, all optimal steering functions u^* are of the form*

$$u^*(t) = \mathrm{sgn}[\eta Y(t)] \tag{5}$$

where η is some n-dimensional vector. (For r-dimensional vectors a and b, $a = \mathrm{sgn}\, b$ means that $a_i = \mathrm{sgn}\, b_i$, $i = 1, \ldots, r$.)

Let $y^j(t)$ be the jth column vector of $Y(t)$. The control system (1) is said to be *normal* if on each interval of positive length and for each $j = 1, \ldots, r$ the functions $y_1^j(t), \ldots, y_n^j(t)$ are linearly independent. This is equivalent to saying that no component of $\eta Y(t)$, $\eta \neq 0$, is identically zero on an interval of positive length, and therefore $u^*(t)$ is uniquely determined by (5). Hence

THEOREM 4. *For normal control systems the general problem has at most one optimal steering function.*

Thus the only way of reaching the objective in minimum time using a normal system is by at all times utilizing properly all of the power available.

3. *The Special Problem.*—The control problem for the system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{6}$$

where the objective is to start at the initial state x and to reach the origin (the equilibrium state) in minimum time will be called *the special problem.* Hence for the special problem we want (see equation (4))

$$-x_0 = \int_0^t Y(\tau)u(\tau)d\tau. \tag{7}$$

It is then not difficult to show that

Theorem 5. *If for some $t > 0$ and some n-vector η there is a solution $u = u^*$ of* (7) *of the form*

$$u^*(\tau) = \operatorname{sgn}[\eta Y(\tau)], \tag{5}$$

then it is an optimal steering function for the special problem.

It is this result that is of interest in solving the synthesis problem. If the control system is autonomous (equation (2)), then we can start the control system at the origin, use a steering function of the form (4) and look at the solution as t decreases (replace t by $-t$). This steering function is then optimal for all the states that can be reached in this manner. Reversing the system in this way gives the set of all initial states in the special problem for which this steering function is optimal. For normal systems the optimal steering is unique, and this procedure always determines the optimal steering as a function of the state of the system. We say "always" in the above sentence because we know that the synthesis problem can be solved in this way for some systems that are not normal. This procedure leads to the determination of switching-surfaces, which are surfaces where certain of the components of the steering change sign.

It is now that we can see the usefulness of introducing another classification of control systems. If $\eta Y(t) \equiv 0$ on any interval of positive length implies $\eta = 0$, then the control system (1) is said to be *proper*. This is equivalent to saying that the row vectors $y_1(t), \ldots, y_n(t)$ of $Y(t)$ are linearly independent vector functions on each interval of positive length. It is clear that *every normal control system is proper* but the converse is not true. It is also not difficult to see, when we remove all constraints on the admissible control functions, that *proper control systems are completely controllable,*[7] i.e., given any two states x_1 and x_2 and any two times t_1 and t_2, $t_1 \neq t_2$, there is a steering function such that starting at x_1 at time t_1 the system is brought to the state x_2 at time t_2.

Proper systems also have the additional controllability property (now we return to the constraint $|u_i(t)| \leq 1$):

THEOREM 6. *If the system (2) is proper and asymptotically stable ($X(t) \rightarrow 0$ as $t \rightarrow \infty$), then for each initial state x_0 there is a steering function in Ω that brings the system to the origin in minimum time.*

It is easy to see for proper systems that optimal steering functions lie on the boundary of Ω. Expressed as a bang-bang principle this states that: *In proper control systems optimal steering* u^* *has the property that at any given time some component of* u^* *is utilizing the maximum power available to it.*

It is of considerable importance to observe that for proper control systems there is a way (if the optimal system for the special problem can be synthesized) of deciding whether or not it is possible to start at a point x_0 and then hit the moving particle $z(t)$ and also possible to determine optimal steering. We can state this result as follows:

For proper control systems the problem $x(0, u) = x_0$, $x(t, u) = z(t)$ for some $t > 0$ and some u in Ω, has a solution if and only if it is possible to start at some point $-w(t_1)$ and then with steering in Ω to reach the origin in time $t_2 \leq t_1$. If $-w(t_1)$ is the first point on the curve $-w(t)$, $t > 0$, from which it is possible to reach in this manner the origin in time t_1, then any steering that does this is optimal for this special problem and is also optimal for the general problem of hitting $z(t)$.

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[4] Krasovskii, N. N., "Concerning the Theory of Optimal Control," *Avtomat. i Telemeh.*, 960–970 **18**, (1957). (Russian.)

[5] Gamkrelidze, R. V., "Theory of Time-Optimal Processes for Linear Systems," *Izvestia Akad. Nauk, SSSR*, **22**, 449–474 (1958). (Russian.)

[6] A simple and elegant proof of this Lemma was pointed out to me by L. Pukanszki using a theorem of Liapunov. See Liapunov, A., "Sur les fonctions-vecteurs completement additives," *Bull. Acad. Sci. URSS Ser. Math. Izvestia Akad. Nauk, SSSR*, **4**, 465–478 (1940); Halmos, P. R., "The Range of a Vector Measure," *Bull. Am. Math. Soc.*, **54**, 416–421 (1948).

[7] The concept of controllability has been introduced by R. E. Kalman and will be discussed by him in a paper to appear.